

# QUOTIENTS OF $\mathbb{N}^*$ , $\omega$ -LIMIT SETS, AND CHAIN TRANSITIVITY

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ABSTRACT.  $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$  has a canonical dynamical structure provided by the shift map, the unique continuous extension to  $\beta\mathbb{N}$  of the map  $n \mapsto n + 1$  on  $\mathbb{N}$ . Here we investigate the question of what dynamical systems can be written as quotients of  $\mathbb{N}^*$ . We prove that a dynamical system is a quotient of  $\mathbb{N}^*$  if and only if it is isomorphic to the  $\omega$ -limit set of some point in some larger system. This provides a full external characterization of the quotients of  $\mathbb{N}^*$ . We also prove, assuming  $\text{MA}_{\sigma\text{-centered}}(\kappa)$ , that a dynamical system of weight  $\kappa$  is a quotient of  $\mathbb{N}^*$  if and only if it is chain transitive. This provides a consistent partial internal characterization of the quotients of  $\mathbb{N}^*$ , and a full internal characterization for metrizable systems.

## 1. INTRODUCTION

$\mathbb{N}^*$  is an important object in (at least) two different categories: the category of topological spaces and the category of dynamical systems. Both of these categories come equipped with an idea of a quotient, and it is a natural problem to characterize the quotients of an important object like  $\mathbb{N}^*$ .

In the category of topological spaces, a “quotient” means a continuous surjection. Much work has been done on the problem of classifying the continuous images of  $\mathbb{N}^*$  (see, e.g., [15], [17], [10], and [12]). An “external” characterization is known:  $X$  is a quotient of  $\mathbb{N}^*$  if and only if it is the remainder of some compactification of  $\mathbb{N}$  (see [11], Theorem 3.5.13). Finding a general internal characterization is more difficult, though many good partial results and consistency results are known. One such is especially relevant here: assuming  $\text{MA}_{\sigma\text{-centered}}(\kappa)$ , every compact space of weight  $\kappa$  is a continuous image of  $\mathbb{N}^*$ .

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In this paper we investigate the problem of classifying the quotients of  $\mathbb{N}^*$  in the category of dynamical systems. Both of the results mentioned in the previous paragraph have analogues in the dynamical setting, and these will constitute our two main theorems. The dynamical version of the external characterization is

**Main Theorem 1.** *Let  $(X, f)$  be any dynamical system.  $(X, f)$  is a quotient of  $\mathbb{N}^*$  if and only if it is isomorphic to the  $\omega$ -limit set of some point in some dynamical system.*

As in the topological category, an internal characterization is more difficult. However, in analogy with the aforementioned topological result, we will prove

**Main Theorem 2.** *Assume  $\text{MA}_{\sigma\text{-centered}}(\kappa)$  and let  $(X, f)$  be a dynamical system where the weight of  $X$  is at most  $\kappa$ . Then  $(X, f)$  is a quotient of  $\mathbb{N}^*$  if and only if  $X$  is chain transitive.*

Both of these theorems are stated here in a slightly simplified form; see Theorems 3.4 and 4.1 for the full statements.

Note that setting  $\kappa = \aleph_0$  in Main Theorem 2 gives a ZFC-provable theorem about metrizable dynamical systems: a metrizable dynamical system is a quotient of  $\mathbb{N}^*$  if and only if it is chain transitive. Very roughly, Main Theorem 2 suggests that the property of chain transitivity somehow captures the important features of the dynamical structure of  $\mathbb{N}^*$ . For more on this idea, see the last section of [9].

Putting our two main theorems together, we obtain:

**Corollary.** *A metrizable dynamical system is chain transitive if and only if it is isomorphic to the  $\omega$ -limit set of some point in some dynamical system.*

For some dynamical systems, the chain transitive subsystems of  $X$  are precisely its  $\omega$ -limit sets. This is true for shifts of finite type (see [3]), topologically hyperbolic maps (see [4]), and certain Julia sets (see [5] and [6]). A good deal of interesting research has been done in the past decade or two on the question of how the chain transitive subsystems of a given system  $X$  correspond to the  $\omega$ -limit sets of  $X$  (see [16], [1], and [2] for further examples). Our corollary here answers a “context-free” version of this question:  $\omega$ -limit sets and chain transitive systems are, up to isomorphism, the same thing.

The structure of the paper is as follows. In Section 2 we will provide a review of some definitions and preliminary results. In Section 3 we will prove our first main theorem, essentially by showing that every dynamical quotient of  $\mathbb{N}^*$  can be expanded to a dynamical quotient of  $\beta\mathbb{N}$ . In Section 4 we will prove our second main theorem.

## 2. DEFINITIONS AND PRELIMINARIES

By a **dynamical system** we mean a compact Hausdorff space  $X$  together with a continuous map  $f : X \rightarrow X$ . A **metrizable system** is a dynamical system with  $X$  metrizable.

Classically, chain transitivity is a property of metrizable systems. However, using the language of uniformities, one can define chain transitivity for non-metrizable systems also, as was done in [9]. If  $\mathcal{U}$  is any open cover of  $X$ , then  $\langle x_0, x_1, \dots, x_n \rangle$  is a  $\mathcal{U}$ -**chain** from  $x_0$  to  $x_n$  provided that, for every  $i < n$ ,  $f(x_i)$  and  $x_{i+1}$  are both in some single member of  $\mathcal{U}$ .  $X$  is **chain transitive** if, for any open cover  $\mathcal{U}$  of  $X$  and any  $x, y \in X$ , there is a  $\mathcal{U}$ -chain from  $x$  to  $y$ . One can easily check that this general definition coincides with the classical one in metrizable systems.

A **subsystem** of  $(X, f)$  is a compact subspace of  $X$  that is closed under  $f$ . For all  $x \in X$ , the  $\omega$ -**limit set** of  $x$ , denoted  $\omega(x)$ , is the set of limit points of the orbit of  $x$ . That is,

$$\omega(x) = \bigcap_{n \in \mathbb{N}} \overline{\{f^m(x) : m \geq n\}}.$$

Every  $\omega$ -limit set in  $X$  is a subsystem of  $X$ , and if  $X$  is metrizable then any  $\omega$ -limit set in  $X$  is chain transitive (see [13] for a proof). We will show below that this is true for non-metrizable systems as well.

$\beta\mathbb{N}$  is the Stone-Čech compactification of the countable discrete space  $\mathbb{N}$ , and we identify the elements of  $\beta\mathbb{N}$  with the ultrafilters on  $\mathbb{N}$ .  $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$  is the space of free ultrafilters on  $\mathbb{N}$ . The topology on  $\beta\mathbb{N}$  is generated by sets of the form  $\overline{A} = \{p \in \beta\mathbb{N} : A \in p\}$ , where  $A \subseteq \mathbb{N}$ , and we write  $A^* = \overline{A} \cap \mathbb{N}^*$ . For more on the topology of  $\beta\mathbb{N}$  and  $\mathbb{N}^*$ , we refer the reader to [15].

If  $X$  is a compact Hausdorff space and  $f : \mathbb{N} \rightarrow X$  is any function, then there is a unique continuous function  $\beta f : \beta\mathbb{N} \rightarrow \beta X$  that extends  $f$ . This function is called the **Stone extension** of  $f$ .

As usual, we write  $A + n$  for  $\{m + n : m \in A\}$ . For each  $p \in \mathbb{N}^*$ , define  $\sigma(p)$  to be the unique ultrafilter generated by  $\{A + 1 : A \in p\}$ . This is called the **shift map** on  $\beta\mathbb{N}$ , and whenever we speak of  $\beta\mathbb{N}$  or  $\mathbb{N}^*$  as a dynamical system it is understood that we are talking about the shift map. The shift map is the Stone extension of the map  $n \mapsto n + 1$ .

For a given  $p \in \beta\mathbb{N}$  and a sequence  $\langle x_n : n \in \mathbb{N} \rangle$  of points in some dynamical system  $X$ , we write  $p\text{-}\lim_{n \in \mathbb{N}} x_n$  for the image of  $p$  under the Stone extension of  $n \mapsto x_n$ . Equivalently,  $p\text{-}\lim_{n \in \mathbb{N}} x_n = y$  if and only if for every open  $U \ni y$  we have  $\{n : x_n \in U\} \in p$ .

**Lemma 2.1.** *Let  $X$  be a compact Hausdorff space and  $\langle x_n : n \in \mathbb{N} \rangle$  a sequence of points in  $X$ .*

- (1)  $p \mapsto p\text{-}\lim_{n \in \mathbb{N}} x_n$  is a continuous map  $\beta\mathbb{N} \rightarrow X$ .
- (2) If  $g : X \rightarrow X$  is continuous and  $p \in \mathbb{N}^*$ , then

$$g\left(p\text{-}\lim_{n \in \mathbb{N}} x_n\right) = p\text{-}\lim_{n \in \mathbb{N}} g(x_n).$$

- (3) For each  $p \in \beta\mathbb{N}$ ,  $p\text{-}\lim_{n \in \mathbb{N}} x_{n+1} = \sigma(p)\text{-}\lim_{n \in \mathbb{N}} x_n$ .

*Proof.* Using the definition of ultrafilter limits as Stone extensions, these are all straightforward. See [8] for more detail, more information, and some interesting discussion.  $\square$

Lemma 2.1(1) will be of special interest to us when the sequence in question is the orbit of some point  $x$ .

Recall that two dynamical systems  $(X, f)$  and  $(Y, g)$  are **isomorphic** (or, for some authors, conjugate), written  $(X, f) \cong (Y, g)$  or  $X \cong Y$ , if there is a homeomorphism  $h : X \rightarrow Y$  such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ h \downarrow & & \downarrow h \\ Y & \xrightarrow{g} & Y \end{array}$$

$Y$  is a **quotient** of  $X$  (or is semi-conjugate to  $X$ ) if  $h : X \rightarrow Y$  is a continuous surjection, but not necessarily a homeomorphism, for which the above diagram commutes. In this case,  $h$  is called a **quotient map** (or a semi-conjugation).

### 3. AN EXTERNAL CHARACTERIZATION OF THE QUOTIENTS OF $\mathbb{N}^*$

The next lemma can be paraphrased as saying that every  $\omega$ -limit set is a quotient of  $\mathbb{N}^*$ . The main result of this section is that the converse also holds: every quotient of  $\mathbb{N}^*$  is an  $\omega$ -limit set.

**Lemma 3.1.** *Let  $(X, f)$  be any dynamical system and  $x \in X$ . The map  $p \mapsto p\text{-}\lim_{n \in \mathbb{N}} f^n(x)$  is a quotient map from  $(\mathbb{N}^*, \sigma)$  onto  $(\omega(x), f)$ .*

*Proof.* This is well-known, and is discussed, e.g., in Section 2 of [8]. The proof is both short and instructive, so we give it here for completeness.

Let  $F_x : \beta\mathbb{N} \rightarrow X$  denote the map  $p \mapsto p\text{-}\lim_{n \in \mathbb{N}} f^n(x)$ . By Lemma 2.1 (1) and (2),  $F_x$  is continuous and maps  $n \in \mathbb{N}$  to  $f^n(x)$ .

Note that  $\beta\mathbb{N} \setminus \{0, \dots, n\}$  is compact and contains  $\mathbb{N} \setminus \{0, \dots, n\}$  as a dense subset. Thus

$$F_x(\beta\mathbb{N} \setminus \{0, \dots, n\}) = \overline{F_x(\mathbb{N} \setminus \{0, \dots, n\})} = \overline{\{f^m(x) : m > n\}}.$$

It follows that

$$F_x(\mathbb{N}^*) = \bigcap_{n \in \mathbb{N}} F_x(\beta\mathbb{N} \setminus \{0, \dots, n\}) = \bigcap_{n \in \mathbb{N}} \overline{\{f^m(x) : m > n\}} = \omega(x),$$

so that  $F_x$  maps  $\mathbb{N}^*$  onto  $\omega(x)$ . Using parts (2) and (3) of Lemma 2.1,

$$\begin{aligned} f(F_x(p)) &= f(p\text{-}\lim_{n \in \mathbb{N}} f^n(x)) = p\text{-}\lim_{n \in \mathbb{N}} f^{n+1}(x) \\ &= \sigma(p)\text{-}\lim_{n \in \mathbb{N}} f^n(x) = F_x(\sigma(p)) \end{aligned}$$

for all  $p \in \beta\mathbb{N}$ . Thus  $F_x$  is a quotient map.  $\square$

Since it is relevant to our investigation here, we point out the following easy corollary. This generalizes results in [13], where the same thing is proved for metrizable systems.

**Corollary 3.2.** *If  $X$  is any dynamical system and  $x \in X$ , then  $\omega(x)$  is chain transitive.*

*Proof.* We have just seen that  $\omega(x)$  is chain transitive. The corollary is now a consequence of Theorem 5.3 and Lemma 5.4 in [9] (Theorem 5.3 states that  $(\mathbb{N}^*, \sigma)$  is chain transitive, and Lemma 5.4 states that quotients of chain transitive systems are again chain transitive).  $\square$

**Lemma 3.3.** *Any quotient of  $\mathbb{N}^*$  can be expanded to a quotient of  $\beta\mathbb{N}$ . More precisely, if  $Q : (\mathbb{N}^*, \sigma) \rightarrow (X, f)$  is a quotient map, then there are some  $\tilde{X}$ ,  $\tilde{f}$ , and  $\tilde{Q}$  such that  $X \subseteq \tilde{X}$ ,  $\tilde{f} \upharpoonright X = f$ , and  $\tilde{Q} : (\beta\mathbb{N}, \sigma) \rightarrow (\tilde{X}, \tilde{f})$  is a quotient map with  $\tilde{Q} \upharpoonright \mathbb{N}^* = Q$ .*

*Proof.* The proof is a modification of the proof of Theorem 3.5.13 in [11], which gives the same theorem for the category of topological spaces (i.e., forgetting  $f$ ,  $\sigma$ , and  $\tilde{f}$ , and replacing “quotient map” with “continuous surjection”).

Let  $X$ ,  $f$ , and  $Q$  be as in the statement of the theorem. Let  $R$  be the equivalence relation on  $\beta\mathbb{N}$  whose equivalence classes are all the singletons from  $\mathbb{N}$  and all the fibers of  $Q$ . In Theorem 3.5.13 of [11], it is proved that this equivalence relation induces a continuous surjection  $\tilde{Q} : \beta\mathbb{N} \rightarrow \tilde{X}$ , where  $\tilde{X} = \beta\mathbb{N}/R$  is a compact Hausdorff space when given the standard quotient topology.

Letting  $[p]$  denote the equivalence class of  $p$ , it is clear (because  $Q$  is a quotient map on  $\mathbb{N}^*$ ) that  $\sigma$  is compatible with  $R$ : i.e.,  $[x] = [y]$  implies  $[\sigma(x)] = [\sigma(y)]$ . Thus we can define  $\tilde{f}$  on  $\tilde{X}$  by  $\tilde{f}([x]) =$

$[\sigma(x)]$ . Identifying  $X$  with  $\tilde{Q}(\mathbb{N}^*)$  in the obvious way, it is clear from our definition of  $\tilde{f}$  that  $\tilde{f} \upharpoonright X = f$ .

If  $U \subseteq \beta\mathbb{N}$  is open and is a union of  $R$ -equivalence classes, then  $\sigma^{-1}(U)$  is again open (because  $\sigma$  is continuous on  $\beta\mathbb{N}$ ) and a union of  $R$ -equivalence classes (because  $\sigma$  maps equivalence classes to equivalence classes). It follows from the definition of a quotient topology that  $\tilde{f}$  is continuous on  $\tilde{X}$ . Thus  $(\tilde{X}, \tilde{f})$  is a dynamical system.

We have seen already that  $\tilde{Q}$  is a continuous surjection, and  $\tilde{Q}$  is a quotient mapping by our definition of  $\tilde{f}$ .  $\square$

**Theorem 3.4** (Main Theorem 1). *Let  $(X, f)$  be any dynamical system.  $X$  is a quotient of  $\mathbb{N}^*$  if and only if there is a dynamical system  $(Y, g)$  and some  $y \in Y$  such that  $X \cong \omega(y)$ . If  $f$  is a homeomorphism, we may assume that  $g$  is also a homeomorphism.*

*Proof.* The first claim of the theorem is mostly proved. Lemma 3.1 gives the “if” direction. For the “only if” direction, find some  $\tilde{X}$ ,  $\tilde{f}$ , and  $\tilde{Q}$  as in the statement of Lemma 3.3. Since  $\omega(0) = \mathbb{N}^*$  in  $\beta\mathbb{N}$ , it is easy to check that  $\omega(\tilde{Q}(0)) = \tilde{Q}(\mathbb{N}^*) = X$  in  $\tilde{X}$ .

For the second claim, we take  $\tilde{X}$ ,  $\tilde{f}$ , and  $\tilde{Q}$  as constructed in the proof of Lemma 3.3. First note that  $f$  is surjective because  $(X, f)$  is a quotient of  $(\mathbb{N}^*, \sigma)$ , and  $\sigma$  is surjective on  $\mathbb{N}^*$ . Thus  $f$  is a homeomorphism if and only if  $f$  is injective (recall that every continuous bijection on a compact Hausdorff space is a homeomorphism). If  $f$  is injective then  $\tilde{f}$  clearly is too, but  $\tilde{f}$  is not surjective. However, there is only one point of  $\tilde{X}$  that lacks a preimage under  $\tilde{f}$ , namely the point  $\tilde{Q}(0)$ . We can remedy the situation by putting  $Y = \tilde{X} \cup (\mathbb{Z} \setminus \mathbb{N}) \cup \{-\infty\}$ , with the points  $(\mathbb{Z} \setminus \mathbb{N}) \cup \{-\infty\}$  having their natural topology as a convergent sequence, and defining  $g : Y \rightarrow Y$  by

$$g(x) = \begin{cases} -\infty & \text{if } x = -\infty \\ n+1 & \text{if } x = n \in (\mathbb{Z} \setminus \mathbb{N}) \setminus \{-1\} \\ \tilde{Q}(0) & \text{if } x = -1 \\ \tilde{f}(x) & \text{if } x \in \tilde{X}. \end{cases}$$

$Y$  is a compact Hausdorff space and  $g$  is a continuous bijection, hence a homeomorphism, on  $Y$ .  $\square$

As promised, this theorem gives us an external characterization of the quotients of  $\mathbb{N}^*$ . Before moving on, we note that the dynamical quotients of  $\beta\mathbb{N}$  have an easy characterization: they are precisely those systems in which some point has a dense orbit. In fact, if the orbit of  $x$  is dense in  $X$  then  $p \mapsto p\text{-}\lim_{n \in \mathbb{N}} f^n(x)$  is a quotient map.

#### 4. CHAIN TRANSITIVITY

In this section we will prove:

**Theorem 4.1** (Main Theorem 2). *Assume  $\text{MA}_{\sigma\text{-centered}}(\kappa)$ , and let  $(X, f)$  be a dynamical system with  $w(X) \leq \kappa$ . The following are equivalent:*

- (1)  *$X$  is chain transitive.*
- (2)  *$X$  is a quotient of  $\mathbb{N}^*$ .*
- (3) *There is some system  $(Y, g)$  with  $w(Y) = w(X)$  and some  $y \in Y$  such that  $\omega(y) \cong X$ .*

*Furthermore, in (3), if  $f$  is a homeomorphism then  $g$  can be taken to be a homeomorphism as well.*

As usual,  $w(X)$  denotes the weight of  $X$ , i.e., the smallest cardinality of a base for  $X$ . We refer the reader to [14] for terminology concerning forcing posets and a definition of  $\text{MA}_{\sigma\text{-centered}}(\kappa)$ . We will point out that, by Bell's Theorem (see [7]), the assumption  $\text{MA}_{\sigma\text{-centered}}(\kappa)$  is equivalent to the assumption  $\mathfrak{p} > \kappa$ .

The equivalence of (1) and (2) is the second main theorem promised in the introduction. The equivalence of (2) and (3) is a strengthening of our first main theorem (it is stronger because we are now able to bound the weight of  $Y$ ).

As an important special case, consider when  $\kappa = \aleph_0$ . It is well-known that  $\text{ZFC} \vdash \text{MA}_{\sigma\text{-centered}}(\aleph_0)$  and that a compact Hausdorff space has countable weight if and only if it is metrizable (by Urysohn's Theorem). Combining these two facts, we obtain:

**Corollary 4.2.** *Let  $(X, f)$  be a metrizable system. The following are equivalent:*

- (1)  *$X$  is chain transitive.*
- (2)  *$X$  is a quotient of  $\mathbb{N}^*$ .*
- (3) *There is some metrizable system  $(Y, g)$  and some  $y \in Y$  such that  $\omega(y) \cong X$ .*

*Furthermore, in (3), if  $f$  is a homeomorphism then  $g$  can be taken to be a homeomorphism as well.*

While the proof of Theorem 4.1 will use Martin's Axiom and the dynamical structure of  $\beta\mathbb{N}$ , it is possible to prove part of Corollary 4.2, namely  $(1) \Leftrightarrow (3)$ , using only a bit of ingenuity and undergraduate-level analysis. We leave this simplification as an exercise for the interested reader.

We now proceed to the proof of Theorem 4.1. That (3) implies (2) is given by Theorem 3.4. That (2) implies (1) is given by Corollary 3.2.

Neither of these implications uses  $\text{MA}_{\sigma\text{-centered}}(\kappa)$ . We will be done once we can prove:

**Theorem 4.3.** *Assume  $\text{MA}_{\sigma\text{-centered}}(\kappa)$ . If  $(X, f)$  is a chain transitive system and  $w(X) = \kappa$ , then there is a dynamical system  $(Y, g)$  with  $w(Y) = \kappa$  and a point  $y \in Y$  such that  $\omega(y) \cong X$ . Furthermore, if  $f$  is a homeomorphism then  $Y$  can be taken to be a homeomorphism as well.*

Note that if  $\lambda \leq \kappa$  then  $\text{MA}_{\sigma\text{-centered}}(\kappa)$  implies  $\text{MA}_{\sigma\text{-centered}}(\lambda)$ . This justifies writing “ $w(X) \leq \kappa$ ” rather than “ $w(X) = \kappa$ ” in the statement of Theorem 4.1.

*Proof of Theorem 4.3.* Assume  $\kappa$  is infinite, since otherwise the theorem is trivial.

Let  $(X, f)$  be a chain transitive system with  $w(X) = \kappa < \mathfrak{c}$ .  $X$  embeds in  $[0, 1]^\kappa$  so, without loss of generality, we may assume that  $X$  is a subset of  $[0, 1]^\kappa$ .  $X$  is closed in  $[0, 1]^\kappa$  because it is compact. Since  $[0, 1]^\kappa$  is homeomorphic to  $[0, 1]^\kappa \times [0, 1]$ , we may assume that  $X$  is nowhere dense in  $[0, 1]^\kappa$  (e.g., if  $X \subseteq [0, 1]^\kappa \times \{0\} \subseteq [0, 1]^\kappa \times [0, 1]$ ).

Recall that  $[0, 1]^\kappa$  is separable (see, e.g., Exercise III.2.13 in [14]), and fix a countable dense  $D \subseteq [0, 1]^\kappa$ . Because  $X$  is nowhere dense in  $[0, 1]^\kappa$ , we may assume that  $X \cap D = \emptyset$ .

The basic idea of the proof is to use Martin’s Axiom to find a sequence  $\langle x_n : n \in \omega \rangle$  of points in  $D$  such that

- $Y = X \cup \{x_n : n \in \omega\}$  is a closed subset of  $[0, 1]^\kappa$ .
- defining  $g(x_n) = x_{n+1}$  and  $g \upharpoonright X = f$ ,  $g$  is continuous on  $Y$ .
- in the dynamical system  $(Y, g)$ ,  $\omega(x_0) = X$ .

Before describing the poset we will use, it will be convenient to make a few definitions. Arbitrarily fix a “base point”  $x \in X$ , and fix a basis  $\mathcal{B}$  for  $[0, 1]^\kappa$  such that  $|\mathcal{B}| = \kappa$ . Without loss of generality, we may assume that if  $U \in \mathcal{B}$  then  $[0, 1]^\kappa \setminus \overline{U}$  is also in  $\mathcal{B}$ . By a *nice cover*, or simply a *cover*, of  $X$  we mean a finite subset  $\mathcal{U}$  of  $\mathcal{B}$  such that every member of  $\mathcal{U}$  meets  $X$  and  $X \subseteq \bigcup \mathcal{U}$ . If  $\mathcal{U}$  and  $\mathcal{V}$  are covers of  $X$ , we say  $\mathcal{V}$  *refines*  $\mathcal{U}$  if every member of  $\mathcal{V}$  is contained in some member of  $\mathcal{U}$ . If  $\mathcal{U}$  is a cover of  $X$  and  $y, z \in [0, 1]^\kappa$ , we say that  $(y, z)$  is  $\mathcal{U}$ -*good* if there is some  $U_y, U_z \in \mathcal{U}$  such that  $y \in U_y$ ,  $z \in U_z$ , and  $f(U_y \cap X) \cap U_z \neq \emptyset$  (compare this with the definition of a chain). If  $\mathcal{U}$  is a cover of  $X$ , we say that a sequence  $\xi = \langle d_i : i \leq n \rangle$  of points in  $D$  is an  $x$ - $\mathcal{U}$ -*loop* if there is some  $U \in \mathcal{U}$  with  $d_0, x \in U$ , if  $(d_i, d_{i+1})$  is  $\mathcal{U}$ -good for every  $i < n$ , and if  $(d_n, x)$  is  $\mathcal{U}$ -good.



Let  $\mathbb{P}$  be the set of all pairs  $\langle s, \mathcal{U} \rangle$ , such that  $s$  is an injective sequence of points in  $D$  and  $\mathcal{U}$  is a cover of  $X$ . We say that  $\langle t, \mathcal{V} \rangle \leq \langle s, \mathcal{U} \rangle$  whenever

- $s$  is an initial segment of  $t$ .
- $\mathcal{V}$  refines  $\mathcal{U}$ .
- $t \setminus s$  is an  $x$ - $\mathcal{U}$ -loop.

Recall that we are trying to build a sequence  $\xi$  of points in  $D$ . Intuitively, a condition  $\langle s, \mathcal{U} \rangle$  is a promise that  $s$  is an initial segment of  $\xi$ , and that  $\xi$  will follow the behavior of  $f$  on  $X$  with a certain “closeness” prescribed by  $\mathcal{U}$ . Roughly speaking, we are building  $\xi$  one loop at a time, using loops from the base point  $x$  that more and more closely approximate the behavior of  $f$ .

The reader may think it strange that we build  $\xi$  by adding loops instead of merely adding elements, which may seem more natural at first glance. But this device is what enables us to prove that  $\mathbb{P}$  is a pre-order (reflexive and transitive):

*Claim.* If  $\langle u, \mathcal{W} \rangle \leq \langle t, \mathcal{V} \rangle \leq \langle s, \mathcal{U} \rangle$  then  $\langle u, \mathcal{W} \rangle \leq \langle s, \mathcal{U} \rangle$ ; i.e.,  $\mathbb{P}$  is transitive.

*Proof.* Let  $\langle u, \mathcal{W} \rangle \leq \langle t, \mathcal{V} \rangle \leq \langle s, \mathcal{U} \rangle$ . The nontrivial thing to check is that  $u \setminus s$  is an  $x$ - $\mathcal{U}$ -loop. But  $u \setminus s$  is the concatenation of the  $x$ - $\mathcal{U}$ -loop  $t \setminus s$  and the  $x$ - $\mathcal{V}$ -loop  $u \setminus t$ . Applying our definition in a straightforward way: since  $\mathcal{V}$  refines  $\mathcal{U}$ ,  $u \setminus t$  is an  $x$ - $\mathcal{U}$ -loop, so  $u \setminus s$  is the concatenation of two  $x$ - $\mathcal{U}$ -loops, and the concatenation of two  $x$ - $\mathcal{U}$ -loops is again an  $x$ - $\mathcal{U}$ -loop.  $\square$

$\mathbb{P}$  is obviously reflexive, so  $\mathbb{P}$  is a pre-order, and can be viewed as a forcing poset with largest element  $(\emptyset, \{[0, 1]^\kappa\})$ . Note that  $\mathbb{P}$  is not antisymmetric: it is possible for  $\mathcal{U}$  to refine  $\mathcal{V}$  and for  $\mathcal{V}$  to refine  $\mathcal{U}$  without having  $\mathcal{U} = \mathcal{V}$ .

*Claim.*  $\mathbb{P}$  is  $\sigma$ -centered.

*Proof.* Because  $D$  is countable, there are only countably many possibilities for the first coordinate of a condition in  $\mathbb{P}$ . To show that  $\mathbb{P}$  is  $\sigma$ -centered, it suffices to show that if two conditions  $\langle s, \mathcal{U} \rangle, \langle s, \mathcal{V} \rangle$  have the same first coordinate  $s$ , then they have a common extension with first coordinate  $s$ . Taking  $\mathcal{W} = \{U \cap V : U \in \mathcal{U} \wedge V \in \mathcal{V} \wedge U \cap V \neq \emptyset\}$ ,  $\langle s, \mathcal{W} \rangle$  is such an extension.  $\square$

If  $s$  is a sequence, let  $R(s)$  denote its range. For each  $U \in \mathcal{B}$  with  $U \cap X \neq \emptyset$ , let

$$A_U = \{\langle s, \mathcal{U} \rangle \in \mathbb{P} : R(s) \cap U \neq \emptyset\},$$

and for each cover  $\mathcal{U}$  of  $X$ , let

$$B_{\mathcal{U}} = \{\langle s, \mathcal{V} \rangle \in \mathbb{P} : \mathcal{V} \text{ refines } \mathcal{U}\}.$$

*Claim.* For each  $U \in \mathcal{B}$  with  $U \cap X \neq \emptyset$ ,  $A_U$  is dense in  $\mathbb{P}$ .

*Proof.* This is where we use the chain transitivity of  $X$ . Fix  $U \in \mathcal{B}$  with  $U \cap X \neq \emptyset$ , and let  $\langle s, \mathcal{U} \rangle \in \mathbb{P}$ . Let  $y \in X \cap U$  and let  $\mathcal{U} \upharpoonright X = \{V \cap X : V \in \mathcal{U}\}$ . Because  $X$  is chain transitive, there is a  $(\mathcal{U} \upharpoonright X)$ -chain in  $X$  from  $x$  to  $x$  that has the point  $y$  in its range (concatenate a chain from  $x$  to  $y$  with a chain from  $y$  to  $x$ ). Call this chain  $t_0$  and write  $t_0 = \langle z_i : i \leq n \rangle$ . For each  $z_i$ , let  $d_i$  be an element of the dense set  $D \setminus (R(s) \cup \{d_j : j < i\})$  such that  $z_i$  and  $d_i$  are both in some common element of  $\mathcal{U}$ ; if  $z_i = y$ , then pick  $d_i$  such that  $d_i$  is also in  $U$ . It is clear that  $t = \langle d_i : i < n \rangle$  is an  $x$ - $\mathcal{U}$ -loop, so we have  $\langle s \smallfrown t, \mathcal{U} \rangle \leq \langle s, \mathcal{U} \rangle$ . By our choice of  $d_i$  when  $z_i = y$ , it is clear that  $\langle s \smallfrown t, \mathcal{U} \rangle \in A_U$ .  $\square$

*Claim.* For each cover  $\mathcal{U}$  of  $X$ ,  $B_{\mathcal{U}}$  is dense in  $\mathbb{P}$ .

*Proof.* Fix a cover  $\mathcal{U}$  and let  $\langle s, \mathcal{V} \rangle \in \mathbb{P}$ . Clearly  $\langle s, \mathcal{U} \rangle \in \mathbb{P}$ , and we have already seen (in the proof that  $\mathbb{P}$  is  $\sigma$ -centered) that any two conditions with the same first coordinate have a common extension. This common extension is in  $B_{\mathcal{U}}$ , so  $B_{\mathcal{U}}$  is dense in  $\mathbb{P}$ .  $\square$

Notice that there are  $\kappa$  sets of the form  $A_U$  and  $\kappa$  sets of the form  $B_{\mathcal{U}}$  (because all open sets are chosen from  $\mathcal{B}$  and  $|\mathcal{B}| = \kappa$ ). Therefore we can use  $\text{MA}_{\sigma\text{-centered}}(\kappa)$  to get a filter  $G$  on  $\mathbb{P}$  that meets every  $A_U$  and every  $B_{\mathcal{U}}$ .

Let  $\xi = \langle x_n : n \in \omega \rangle$  be the sequence determined by  $G$ : i.e., we set  $\xi = \bigcup \{s : \langle s, \mathcal{U} \rangle \in G\}$ . Note that  $\xi$  is infinite: if  $s$  is any finite sequence, then the fact that  $G \cap A_{[0,1]^\kappa \setminus R(s)} \neq \emptyset$  implies that  $\xi \neq s$ . Let  $Y = X \cup \{x_n : n \in \omega\}$ , with  $g : Y \rightarrow Y$  defined by  $g \upharpoonright X = f$  and  $g(x_n) = x_{n+1}$ . This map is well-defined because  $R(\xi)$  is infinite, disjoint from  $X$  ( $R(\xi) \subseteq D$  and  $D \cap X = \emptyset$ ), and  $x_n \neq x_m$  whenever  $n \neq m$  (i.e.,  $\xi$  is injective). We claim that  $(Y, g)$  is a dynamical system and that  $\omega(x_0) = X$ .

*Claim.*  $Y$  is compact.

*Proof.* Let  $U$  be an open subset of  $[0,1]^\kappa$  containing  $X$ . There is a collection  $\mathcal{U} \subseteq \mathcal{B}$  with  $X \subseteq \bigcup \mathcal{U} \subseteq U$ . Since  $G \cap B_{\mathcal{U}} \neq \emptyset$  and  $G$  is upward-closed, there is some  $s \in D^{<\omega}$  with  $\langle s, \mathcal{U} \rangle \in G$ . Then, since  $G$  is a filter, if  $\langle t, \mathcal{V} \rangle \in G$  and  $t$  extends  $s$ , we must have  $t \setminus s$  be an  $x$ - $\mathcal{U}$ -loop, and in particular  $R(t \setminus s) \subseteq \bigcup \mathcal{U}$ . This shows that  $Y \setminus U$  is contained in the range of  $s$ , hence finite. Thus  $X$  is a compact subset of  $Y$  with the property that any open set containing  $X$  has finite complement (in  $Y$ ). It follows that  $Y$  is compact.  $\square$

*Claim.*  $X = \{p\text{-}\lim_{n \in \mathbb{N}} x_n : p \in \mathbb{N}^*\}$ .

*Proof.* If  $U$  is any neighborhood of  $X$  then, by the previous paragraph,  $U \cap \{x_n : n \in \mathbb{N}\}$  is cofinite. Thus  $p\text{-}\lim_{n \in \mathbb{N}} x_n \in U$  for any  $p \in \mathbb{N}^*$ , which shows  $\{p\text{-}\lim_{n \in \mathbb{N}} x_n : p \in \mathbb{N}^*\} \subseteq U$ . Since  $U$  was an arbitrary neighborhood of  $X$ ,  $\{p\text{-}\lim_{n \in \mathbb{N}} x_n : p \in \mathbb{N}^*\} \subseteq X$ .

For the opposite inclusion, fix  $U \in \mathcal{B}$  with  $U \cap X \neq \emptyset$ , and let  $s$  be any initial segment of  $\xi$ . Let  $U' = U \setminus R(s)$ . As  $G \cap A_{U'} \neq \emptyset$ , there is some  $\langle t, \mathcal{V} \rangle \in G$  with  $R(t) \cap U' \neq \emptyset$ . In fact,  $t$  is an initial segment of  $\xi$  and, given that  $U'$  is disjoint from  $R(s)$ ,  $t$  must extend  $s$  and a point of  $R(t \setminus s)$  must lie in  $U$ . Since  $s$  was an arbitrary initial segment of  $\xi$ , this shows that infinitely many members of  $R(\xi)$  lie in  $U$ . Letting  $p$  be any member of  $\mathbb{N}^*$  with  $\{n \in \mathbb{N} : x_n \in U\} \in p$ , we have  $p\text{-}\lim_{n \in \mathbb{N}} x_n \in U$ . Since we already know  $p\text{-}\lim_{n \in \mathbb{N}} x_n \in X$  by the previous paragraph, we have  $p\text{-}\lim_{n \in \mathbb{N}} x_n \in U \cap X$ . Since  $U$  was arbitrary, this shows  $\{p\text{-}\lim_{n \in \mathbb{N}} x_n : p \in \mathbb{N}^*\}$  is dense in  $X$ . Since  $\{p\text{-}\lim_{n \in \mathbb{N}} x_n : p \in \mathbb{N}^*\}$  is closed (it is the image of  $\mathbb{N}^*$  under a continuous function), we have  $\{p\text{-}\lim_{n \in \mathbb{N}} x_n : p \in \mathbb{N}^*\} = X$ .  $\square$

We will show next that  $g$  is continuous on  $Y$ , and then, by Lemma 3.1 and the previous claim, we will have  $\omega(x_0) = \{p\text{-}\lim_{n \in \mathbb{N}} x_n : p \in \mathbb{N}^*\} = X$ . Thus, to complete the proof of the first assertion of theorem, it remains to show:

*Claim.*  $g$  is continuous on  $Y$ .

*Proof.* Let  $y, z \in Y$  with  $g(y) = z$ , and let  $U \in \mathcal{B}$  with  $z \in U$ . We will find a  $V \in \mathcal{B}$  such that  $y \in V$  and  $g(Y \cap V) \subseteq U$ . If  $y \in Y \setminus X$  then  $y$  is an isolated point in  $Y$ ; this follows directly from our argument showing  $Y$  is compact. In this case, there is a  $V \in \mathcal{B}$  with  $Y \cap V = \{y\}$ , in which case it is trivially true that  $g(Y \cap V) \subseteq U$ . Thus we may assume  $y \in X$  and  $f(y) = z$ .

Roughly, the case  $y, z \in X$  could only fail if the points of  $Y \setminus X$  make  $g$  discontinuous, i.e., if the behavior of the  $x_n$  near  $X$  does not match the behavior of  $f$  on  $X$ . But this cannot happen, because  $\mathbb{P}$  is designed precisely so that the  $x_n$  will match the behavior of  $X$  to any prescribed level of accuracy.

To make this argument rigorous, we need one more definition. If  $\mathcal{U}$  and  $\mathcal{V}$  are both open covers of a space  $Z$ , recall that  $\mathcal{V}$  is a *star refinement* of  $\mathcal{U}$  if for every  $V \in \mathcal{V}$  there is some  $U \in \mathcal{U}$  such that  $\text{st}(V, \mathcal{V}) = \bigcup \{W \in \mathcal{V} : W \cap V \neq \emptyset\} \subseteq U$ . This definition adapts in the obvious way to nice covers of  $X$ . It is a standard result about star refinements that every (locally) finite open cover of a normal space has a (locally) finite star refinement. The standard proof of this adapts

easily to show that for every nice cover  $\mathcal{U}$  there is a nice cover  $\mathcal{V}$  that is a star refinement of  $\mathcal{U}$ .

*sub-claim.* For every cover  $\mathcal{U}$  of  $X$ , every  $w \in X$ , and every  $W \in \mathcal{B}$  with  $w \in W$ , there is a cover  $\mathcal{V}$  of  $X$  that refines  $\mathcal{U}$  and has the property that  $w$  is in precisely one member of  $\mathcal{V}$ , and, furthermore, that this member of  $\mathcal{V}$  is contained in  $W$ .

*proof of sub-claim.* Let  $\mathcal{U}$  be an  $x$ -cover,  $w \in X$ , and  $w \in W \in \mathcal{B}$ . Pick any  $U \in \mathcal{U}$  with  $w \in U$  and, using the regularity of  $[0, 1]^\kappa$ , pick  $V, V' \in \mathcal{B}$  with  $w \in V' \subseteq \overline{V'} \subseteq V \subseteq U \cap W$ . Let  $\mathcal{V} = \{V\} \cup \{U \setminus \overline{V'} : U \in \mathcal{U}\}$ .  $\square$

Let us now return to the proof of the continuity of  $g$ . We have  $y, z \in X$  with  $f(y) = z$ , and  $U \in \mathcal{B}$  with  $z \in U$ . We would like to find some  $V \in \mathcal{B}$  with  $y \in V$  such that  $g(V \cap Y) \subseteq U$ .

To begin, use our sub-claim to find a cover  $\mathcal{U}_0$  of  $X$  such that  $z$  is contained in a unique member  $U_0$  of  $\mathcal{U}_0$  and  $U_0 \subseteq U$ . Let  $\mathcal{U}$  be a star refinement of  $\mathcal{U}_0$ , and let  $\tilde{U}$  be a member of  $\mathcal{U}$  with  $z \in \tilde{U}$ . Using our sub-claim again, let  $\mathcal{V}_0$  be a refinement of  $\mathcal{U}$  such that only one member  $V_0$  of  $\mathcal{V}_0$  contains  $y$ , and  $V_0 \cap X \subseteq f^{-1}(\tilde{U} \cap X)$ . Let  $\mathcal{V}$  be a star refinement of  $\mathcal{V}_0$  and, using our sub-claim a third time, we may assume that  $y$  is contained in a unique member  $\tilde{V}$  of  $\mathcal{V}$ . By the genericity of  $B_{\mathcal{V}}$ , there is some initial segment  $s$  of  $\xi$  such that  $\langle s, \mathcal{V} \rangle \in G$ . Let  $V = \tilde{V} \setminus R(s)$ .

Suppose  $x_n \in V$ . Because  $\langle s, \mathcal{V} \rangle \in G$ , we must have  $(x_n, x_{n+1})$  be  $\mathcal{V}$ -good. If  $W \in \mathcal{V}$  with  $x_n \in W$ , we must have  $W \subseteq V_0$  (because  $\mathcal{V}$  is a star refinement of  $\mathcal{V}_0$ , and  $V_0$  is the only element of  $\mathcal{V}_0$  containing  $y$ ). In particular,  $W \cap X \subseteq f^{-1}(\tilde{U} \cap X)$ . Because  $(x_n, x_{n+1})$  is  $\mathcal{V}$ -good, this means that, if  $W'$  is any member of  $\mathcal{V}$  containing  $x_{n+1}$ , we must have  $W' \cap \tilde{U} \cap X \neq \emptyset$ . But  $\mathcal{U}$  was defined so that any member of  $\mathcal{U}$  (and, a fortiori, of its refinement  $\mathcal{V}$ ) meeting  $\tilde{U}$  must also be a subset of  $U_0$ , which in turn is a subset of  $U$ . Thus  $W' \subseteq U$  and, in particular,  $x_{n+1} \in U$ .

Now let  $w$  be any member of  $V \cap Y$ . By the previous paragraph, if  $w \in Y \setminus X$  then  $g(w) \in U$ . Because  $V \cap X \subseteq V_0 \cap X \subseteq f^{-1}(\tilde{U} \cap X) \subseteq f^{-1}(U \cap X)$ , if  $w \in X$  then  $g(w) = f(w) \in U$ . Thus  $V$  is a neighborhood of  $y$  with  $g(V \cap Y) \subseteq U$ . This completes the proof that  $g$  is continuous.  $\square$

This completes the proof of the first assertion of the theorem. The second assertion is handled exactly as in the proof of Theorem 3.4.  $\square$

Two natural questions arise upon reading Theorem 4.1. The first is whether (1) – (3) are equivalent in general, without any assumptions on  $X$  or any axiomatic assumptions beyond ZFC. They are not. To see why, notice that if  $X$  is connected then  $(X, \text{id}_X)$  is chain transitive. There are compact connected spaces of arbitrarily high cardinality (e.g., those of the form  $[0, 1]^\kappa$ ), but every quotient of  $\mathbb{N}^*$  has cardinality at most that of  $\mathbb{N}^*$ , namely  $2^{2^{\aleph_0}}$ . Thus we must somehow restrict the size of  $X$  for our theorem to hold – at the very least,  $X$  must be a continuous image of  $\mathbb{N}^*$ .

The second question is whether this is the only restriction necessary beyond chain transitivity. It is obvious that if  $X$  is not a topological quotient of  $\mathbb{N}^*$  then  $(X, f)$  is not a dynamical quotient of  $(\mathbb{N}^*, \sigma)$ , and we know from [9] that every quotient of  $\mathbb{N}^*$  must be chain transitive. So our question is

**Question 4.4.** *If  $X$  is a continuous image of  $\mathbb{N}^*$  and  $f$  is a chain transitive map on  $X$ , is  $(X, f)$  a quotient of  $(\mathbb{N}^*, \sigma)$ ?*

If the answer to this question is yes, then the problem of characterizing the dynamical quotients of  $\mathbb{N}^*$  will be reduced to the well-studied problem of characterizing its continuous images. At the time of writing, not even a consistent answer is known.

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